Replacement 23 - 36

Sunday, 9 October 2022 15:29

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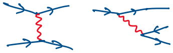
23 • We want to precisely calculate decay rates and scattering probabilities



• To do so, we use **FERMI'S GOLDEN RULE**

$$\boxed{\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \rho(E_f)} \quad Ⓐ$$

which calculates:

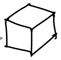
number of transitions per unit time

Γ_{fi} = Rate at which initial state $|i\rangle$ transitions to final state $|f\rangle$

assuming that:

- States are normalised such that

$$\langle i|j\rangle = \int_V \psi_i^* \psi_j \, dV = \delta_{ij}$$



(where V is a box of volume V inside which the states are confined)

- T_{fi} is the "Transition Matrix Element"

$$\text{(A2)} \quad T_{fi} = \langle f|\hat{H}_{int}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}_{int}|j\rangle\langle j|\hat{H}_{int}|i\rangle}{E_i - E_j} + \dots$$

- $\hat{H}_{int}$ is the "interaction" part of the full Hamiltonian:

$$\hat{H}_{full} = \hat{H}_{free} + \hat{H}_{int},$$ and

- $\rho(E_f)$ is the density of final states, defined by

$$\rho(E_f) = \left|\frac{dN}{dE}\right|_{E=E_f}$$

(see revision in next slide)

where dN is the number of final states with energy between E_f and $E_f + dE$

(E_f = Final state energy = Initial State Energy)

ASIDE

A proof of Fermi's Golden Rule in the above form is given in sec 2·3·6 of Mark Thomson's book. It is assumed that you have seen similar proofs in earlier courses, so the origin of F.G.R. is not discussed here. For an alternative more in-depth proof see section 46 of P.A.M. Dirac's book "The Principles of Quantum Mechanics".

Learning/take home points: F.G.R.

$$\boxed{\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \rho(E_f)} \quad Ⓐ$$

Reaction rates are controlled by Matrix Element and Phase Space

Quantum Physics/ Particle Content/ SM/SUSY/Leptoquarks

Relativity, Mom Cons, Energy Cons.

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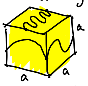
24 Revision of Phase Space (non-relativistic, single particle)

When a state is confined to a box, allowed momenta are only able to take values on a lattice.

Suppose V is a cube of side "a". $(a^3 = V)$



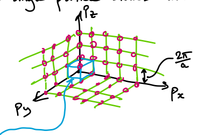
then allowed states $\psi \propto e^{\pm i(\underline{p}\cdot\underline{x} - \omega t)}$ must have $\underline{p}=(p_x,p_y,p_z)=\frac{2\pi}{a}(n_x,n_y,n_z)$

integers

so that periodic B.C.s are satisfied

$$\psi(x+a,y\ \ ,z\ \ )=\psi(x,y,z)$$
$$\psi(x\ \ ,y+a,z\ \ )=\psi(x,y,z)$$
$$\psi(x\ \ ,y\ \ ,z+a)=\psi(x,y,z).$$

Hence single particle states live on this lattice



(Volume of SINGLE STATE in mom space) =

$$=\left(\frac{2\pi}{a}\right)\left(\frac{2\pi}{a}\right)\left(\frac{2\pi}{a}\right)=\frac{(2\pi)^3}{V}$$

∴ number of states dn in element $d^3\underline{p}$ is

$$\boxed{dn=\frac{d^3\underline{p}}{\left(\frac{(2\pi)^3}{V}\right)}=V\frac{d^3\underline{p}}{(2\pi)^3}}\qquad Ⓑ$$

(NOT MANIFESTLY LORENTZ COVARIANT)

ASIDE ON YUKKY PHASE-SPACE

We want $\rho(E_f)\equiv\left|\frac{dn}{dE}\right|_{E=E_f}$.

We COULD get $\rho(E_f)$ by using

① $dn=V\frac{d^3p}{(2\pi)^3}=V\frac{4\pi p^2dp}{(2\pi)^3}\Rightarrow\boxed{\frac{dn}{dp}=V\frac{4\pi p^2}{(2\pi)^3}}$

② $\beta=\frac{p}{E}\Rightarrow p=\beta E\Rightarrow\boxed{\frac{dp}{dE}=\beta}$

to deduce that

$$\rho(E_f)\equiv\left|\frac{dn}{dE}\right|_{E=E}=\left|\frac{dn}{dp}\frac{dp}{dE}\right|_{E=E_f}$$

$$\therefore\ \rho(E_f)=V\frac{4\pi p^2}{(2\pi)^3}\beta$$

but this is SUPER UGLY!

See nice trick on next slide that avoids β usage!

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26 Writing Golden Rule in nicer form.

Ideally we'd like to write the Golden Rule in a form that avoids it being polluted by β's and other things that are frame dependent (except where absolutely necessary). Want to aim for explicit "Lorentz Covariance".

Step 1: write density of states more nicely! ☺.

$$\rho(E_f)=\left|\frac{dn}{dE}\right|_{E_f}=\int\frac{dn}{dE}\delta(E-E_i)\,dE\qquad Ⓒ$$

Why is this better?

- Not because $E_f\to E_i$ ... that's just Energy Conservation.
- Better because integral scans "ALL ENERGIES" rather than "just one".
  - "ALL ENERGIES" ⟶ "ALL ENERGIES" in a boost.
  - "ALL ENERGIES" is a "nice set" under Lorentz Tfms.

What does F.G.R. look like now?

$$\boxed{\Gamma_{if} = 2\pi \, |T_{fi}|^2 \rho(E_f)} \quad Ⓐ$$

using Ⓒ

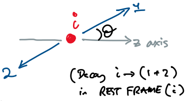
$$\Rightarrow \Gamma_{if} = 2\pi \int |T_{fi}|^2 \frac{dn}{dE} \delta(E-E_i)\, dE$$

$$\Rightarrow \boxed{\Gamma_{if} = 2\pi \int |T_{fi}|^2 \delta(E-E_i)\, dn} \quad Ⓓ$$

NICER

Integral is now over all allowed final states of any energy. ☺

Let's consider a special case:



(Decay $i \to (1+2)$ in REST FRAME (i))

Momentum conservation says that final state $(1+2)$ can be parameterised by the momentum of just one of the two final state particles (w.l.o.g particle 1) since other one has momentum equal and opposite*.

Thus it suffices to replace $dn \to V\frac{d^3p_1}{(2\pi)^3}$ (uses Ⓑ) and since also $E \equiv E_f \equiv E_1 + E_2$ we find that:

* in QFT course you will see this is not an ad-hoc additional principle. It is unavoidable.

$$\boxed{\Gamma_{if} = 2\pi \int |T_{fi}|^2 \delta(E-E_i)\, dn} \quad Ⓓ$$

GENERAL

$$\Rightarrow \boxed{\Gamma_{if} = 2\pi \int |T_{fi}|^2 \delta(E_1+E_2-E_i)\, V\frac{d^3p_1}{(2\pi)^3}} \quad \text{(uses Ⓑ)}$$

$i \to 1+2$

"Because we can!"

$i \to 1+2$

$$\Rightarrow \boxed{\Gamma_{if} = \underbrace{(2\pi)^4}_{\text{MATHS}} \cdot V \cdot \int \underbrace{|T_{fi}|^2}_{\text{PHYSICS}} \underbrace{\delta(E_1+E_2-E_i)}_{\text{ENERGY CONS}} \underbrace{\delta^3(\underline{p}_1+\underline{p}_2-\underline{p}_i)}_{\text{MOM CONS}} \underbrace{\frac{d^3\underline{p}_1}{(2\pi)^3}\frac{d^3\underline{p}_2}{(2\pi)^3}}_{\text{DENSITY OF STATES}}} \quad Ⓔ$$

Nicer Still!

Integral is now over all final state momenta, not just some special ones! ☺

Becoming "more" Lorentz Covariant

Meaning is starting to "factorise".

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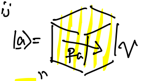
27 Let's try to move to a "better" wave func (or state) normalisation.

("better" = more manifestly Lorentz Covariant)

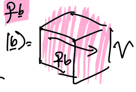
Up to now, we have been using single particle states satisfying

$$\int_V \psi^* \psi \, dV = 1 \quad \text{with} \quad \langle i | j \rangle = \delta_{ij}$$



ie $|\underline{a}\rangle$ = "The box of volume V contains a single particle with energy E_a and mom $\underline{p}_a$"

$|\underline{b}\rangle$ = "The box of volume V contains a single particle with energy E_b and mom $\underline{p}_b$"

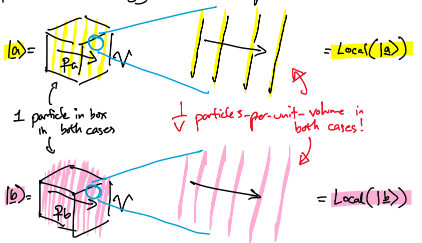


EQUIVALENTLY:

Within a "small bit" of the box, $|a\rangle$ & $|b\rangle$ each look locally like a bit of a

"$\frac{1}{V}$-particles-per-unit-volume soup"

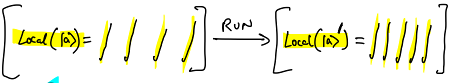
of particles with energy E_a or E_b respectively.



$|a\rangle =$
p_a
V
$= Local(|a\rangle)$
1 particle in box in both cases
$\frac{1}{V}$ particles-per-unit-volume in both cases!
$|b\rangle =$
p_b
V
$= Local(|b\rangle)$

LENGTH CONTRACTION caused by RUNNING PAST

Local regions of $|a\rangle$ changes wavelength and thus also momentum, as expected by Lorentz Tfm:



Alas ☹ ! If we run past $|a\rangle$ at the right rate so as to exactly transform $E_a \to E_a' = E_b$ and $p_a \to p_a' = p_b$

we do **NOT** find that $|a\rangle \to |a\rangle' = |b\rangle$

Instead we find that $|a\rangle' = \sqrt{\gamma}\,|b\rangle$.

$\uparrow$ γ = gamma-factor associated with the boost belonging to our "run"

Why?

Because the LENGTH CONTRACTION also increases the local particle density by a factor of γ! ☹.

[$\int_V \psi^* \psi \, dV = 1$ and $\langle i|j\rangle = \delta_{ij}$, so $\frac{1}{V}$ particles per unit volume as 1 particle in box of volume V.]

Summary:

- One-particle states which are correctly normalised for one observer (according to our current convention) will not appear correctly normalised to observers moving at other velocities.

Choose, therefore, to adopt a new normalization convention which (unlike the last one) is consistent for all observers.

$$\psi_a'' \underset{def}{\equiv} \sqrt{\gamma_a}\,\psi_a$$ could work, and might be elegant,

$\uparrow$ $\gamma_a = \frac{1}{\sqrt{1-\beta_a^2}}$ where $\beta_a = \frac{p_a}{E_a}$

but for a mix of historical and practical reasons we use

$$\psi_a' \underset{def}{\equiv} \boxed{\sqrt{2E_a}\sqrt{V}\,\psi_a \quad Ⓕ}$$

(States in new norm convention are notated with a "prime": ψ_a')

which is called "2E particles per unit volume" since

$$(\#\text{particles in box}) \sim \int_V \psi_a'^* \psi_a' \, dV = 2E_a V \int_V \psi_a^* \psi_a \, dV = 2E_a V \quad \text{and}$$

$$(\text{vol of box}) = V \qquad \therefore \quad (\#\text{particles per unit vol}) \sim \frac{2E_a V}{V} = 2E_a.$$

Where do we use these newly normalised states?

- Recall definition of T_{fi} in Ⓐ2 as a sandwhich of $\hat{H}_{int}$ with "old" states as the bread on either side.

$$\text{Eg:} \qquad T_{fi} = \langle \underbrace{\psi_1, \psi_2, \ldots}_{\langle f|} | \hat{H}_{int} | \underbrace{\ldots \psi_{N-1}, \psi_N}_{|i\rangle} \rangle + \ldots \qquad \text{Ⓖ}$$

- For any such T_{fi}, denote by "M_{fi}" the corresponding

"Lorentz Invariant Matrix Element"

by replacing $\langle f|$ and $|i\rangle$ with states of the new norm (and by multiplying by one conventional factor of $\frac{1}{V}$).

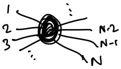
- i.e. for the T_{fi} in Ⓖ define the corresponding M_{fi} to be:

$$\boxed{M_{fi} \underset{def}{\equiv} \frac{1}{V} \langle \psi_1', \psi_2', \ldots | \hat{H}_{int} | \ldots \psi_{N-1}', \psi_N' \rangle + \ldots \quad {\scriptstyle \text{DEFENITION}}} \qquad \text{Ⓗ}$$

$$\therefore \quad M_{fi} = \frac{1}{V} \sqrt{2E_1}\sqrt{V} \ldots \sqrt{2E_N}\sqrt{V} \langle \psi_1, \psi_2 \ldots | \hat{H}_{int} | \ldots \psi_{N-1}, \psi_N \rangle + \ldots$$

COROLLARY:

$$\therefore \quad \boxed{M_{fi} = V^{N/2-1} \left(\sqrt{2E_1} \cdot \ldots \cdot \sqrt{2E_N} \right) T_{fi} . \quad {\scriptstyle \text{COROLLARY}}} \qquad \text{Ⓘ}$$



ASIDE: The benefits of the const factor $\frac{1}{V}$ will be seen later. Likewise, the appropriateness of the name "Lorentz Invariant Matrix Element" is not yet clear, but will later become so.

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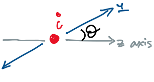
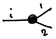
28 Making Ⓔ nicer still.



Recall that for the process we had the expression:

$$\boxed{\Gamma_{if}^{\,i\to 1+2} = \underbrace{(2\pi)^4 \cdot V \cdot \int}_{\text{MATHS}} \underbrace{|T_{fi}|^2}_{\text{PHYSICS}} \underbrace{\delta(E_1+E_2-E_i)}_{\text{ENERGY CONS}} \underbrace{\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_i)}_{\text{MOM CONS}} \underbrace{\frac{d^3\vec{p}_1}{(2\pi)^3} \frac{d^3\vec{p}_2}{(2\pi)^3}}_{\text{DENSITY OF STATES}}} \qquad \text{Ⓔ}$$

Does this become nicer with the newly normalized states?

For this process $N=3$ (i.e. three external legs ) and so

$$\text{Ⓘ} \Rightarrow \qquad M_{fi} = V^{\frac{3}{2}-1} \left(\sqrt{2E_i}\sqrt{2E_1}\sqrt{2E_2} \right) T_{fi}$$

and thus Ⓔ $\Rightarrow$

$$\Gamma_{if} = (2\pi)^4 \cdot \cancel{V} \cdot \int |M_{fi}|^2 \frac{\cancel{V^{2(1-\frac{3}{2})}}}{(2E_i)(2E_1)(2E_2)} \delta(E_1+E_2-E_i)\, \delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_i) \frac{d^3\vec{p}_1}{(2\pi)^3} \frac{d^3\vec{p}_2}{(2\pi)^3}$$

$$\Rightarrow \boxed{\Gamma_{if}^{\,i\to 1+2} = \underbrace{\frac{(2\pi)^4}{2E_i}}_{\substack{\text{TIME} \\ \text{DILATION} \\ \text{FACTOR}}} \int \underbrace{|M_{fi}|^2}_{\text{PHYSICS}} \underbrace{\delta(E_1+E_2-E_i)}_{\text{ENERGY CONS}} \underbrace{\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_i)}_{\text{MOM CONS}} \underbrace{\frac{d^3\vec{p}_1}{(2\pi)^3\, 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3\, 2E_2}}_{\text{DENSITY OF STATES}}} \qquad \text{Ⓙ}$$

SUPER NICE

LHS **SHOULD** time dilate by factor γ of initial state E_i. But $E_i = \gamma m_i$, so time dilation is perfectly captured by leading $\frac{1}{2E_i}$ term! Great! ☺

Separately:

- $\frac{d^3p_1}{(2\pi)^3 2E_1}$ is Lorentz **INVARIANT** (see ExSheet Q2).
- $\delta^4(p_1^\mu + p_2^\mu - p_i^\mu)$ is Lorentz **INVARIANT**.

(non-trivial! uses $\mathrm{Det}(\Lambda) = 1$ and $\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$ in Jacobian)

So BY PROCESS OF ELIMINATION $|M_{fi}|^2$ is also **LORENTZ INVARIANT**.*

* This justifies the name we gave it earlier, and the extra factor of $\frac{1}{V}$ we used in its definition.

ASIDE

At this point one might wish to compare/contrast the calculation leading to ⑤ above for $\boxed{i\to 1+2}$ with a similar but subtly different calculation (page 34) for the case $\boxed{1+2\to 3+4}$. Differences occur because the number of external particles is then N=4 instead of N=3.

——— However ———

Instead we try to use ⑤ to calculate a $\boxed{i\to 1+2}$ particle decay rate.

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Decay Rate Calculations

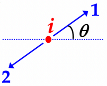
$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

★ **Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient**

- **In the C.o.M. frame** $E_i = m_i$ **and** $\vec{p}_i = 0$ ⇨

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$

- **Integrating over** $\vec{p}_2$ **using the δ-function:**



$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\frac{d^3\vec{p}_1}{4E_1E_2}$$

now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ **since the δ-function imposes** $\vec{p}_2 = -\vec{p}_1$

- **Writing** $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$

For convenience, here $|\vec{p}_1|$ is written as p_1

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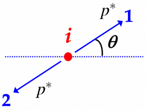
- **Which can be written in the form** $$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 g(p_1)\delta(f(p_1))dp_1 d\Omega \quad (2)$$

where $g(p_1) = p_1^2/(E_1E_2) = p_1^2(m_1^2 + p_1^2)^{-1/2}(m_2^2 + p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2 + p_1^2)^{1/2} - (m_2^2 + p_1^2)^{1/2}$



Note:
- $\delta(f(p_1))$ **imposes energy conservation.**
- $f(p_1) = 0$ **determines the C.o.M momenta of the two decay products**

i.e. $f(p_1) = 0$ **for** $p_1 = p^*$

★ **Eq. (2) can be integrated using the property of δ – function derived earlier (eq. (1))**

$$\int g(p_1)\delta(f(p_1))dp_1 = \frac{1}{|df/dp_1|_{p^*}}\int g(p_1)\delta(p_1 - p^*)dp_1 = \frac{g(p^*)}{|df/dp_1|_{p^*}}$$

where p^* **is the value for which** $f(p^*) = 0$

- **All that remains is to evaluate** df/dp_1

$$\frac{df}{dp_1} = -\frac{p_1}{(m_1^2 + p_1^2)^{1/2}} - \frac{p_1}{(m_2^2 + p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1 + E_2}{E_1E_2}$$

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giving:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{E_1E_2}{p_1(E_1 + E_2)}\frac{p_1^2}{E_1E_2}\right|_{p_1 = p^*} d\Omega$$

$$= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{p_1}{E_1 + E_2}\right|_{p_1 = p^*} d\Omega$$

- **But from** $f(p_1) = 0$, **i.e. energy conservation:** $E_1 + E_2 = m_i$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 d\Omega$$

In the particle's rest frame $E_i = m_i$

$$\Rightarrow \boxed{\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 d\Omega} \quad (3)$$

VALID FOR ALL TWO-BODY DECAYS !

- p^* **can be obtained from** $f(p_1) = 0$

$$(m_1^2 + p^{*2})^{1/2} + (m_2^2 + p^{*2})^{1/2} = m_i \quad \text{(Question 3)}$$

$$\Rightarrow p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2 - (m_1 + m_2)^2\right]\left[m_i^2 - (m_1 - m_2)^2\right]}$$

(now try Questions 4 & 5)

32

Cross section definition

FOR COLLISIONS.

$$\boxed{\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}}$$

Flux = number of incident particles/ unit area/unit time

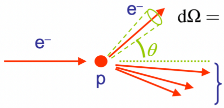
- **The "cross section", σ, can be thought of as the effective cross-sectional area of the target particles for the interaction to occur.**
- **In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption**

here σ is the projective area of nucleus

Differential Cross section

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}}$$

or generally $\frac{d\sigma}{d\ldots}$



$d\Omega = d(\cos\theta)d\phi$

integrate over all other particles

with $\boxed{\sigma = \int \frac{d\sigma}{d\Omega} d\Omega}$

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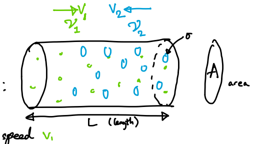
33 Classical link between cross section and effective area of interaction.

Suppose that:

- two beams of particle are in collision:



- "Species 1" (scatterers) are:
  - point-like
  - travelling → with speed v_1
  - have number-density ν_1 particles-per-unit volume
- "Species 2" (targets) are:
  - present area "σ" to species 1.
  - travelling ← with speed v_2
  - have number-density ν_2 particles-per-unit volume

∴ In time δt each scatterer "passes" a volume $(v_1+v_2)\delta t A$ of targets.

∴ 〃 〃 〃 〃 〃 〃 〃 an effective target area of $(v_1+v_2)\delta t A \nu_2 \sigma$ distributed within a total area A.

The probability of interaction of the scatter is thus $(v_1+v_2)\delta t \, \nu_2 \sigma$.

There are, however, $AL\nu_1$ scatterers in the volume, so the expected number of interactions in the whole region is $(v_1+v_2)\delta t \, \nu_2 \sigma A L \nu_1$.

Therefore the interaction rate in this cylinder is $(v_1+v_2)\nu_2\sigma A L \nu_1$ Ⓚ

and the interaction rate per unit volume is $(v_1+v_2)\nu_2\sigma\nu_1$. Ⓛ

Note that Fermi's Golden Rule computes an interaction rate in the cylinder for the special case: $\nu_1 = \frac{1}{V} = \nu_2$ and $AL=V$.

∴ Ⓚ ⇒ $\Gamma_{if} = (v_1 + v_2)\frac{1}{V}\sigma V \frac{1}{V}$

⇒ $\boxed{\sigma = \frac{\Gamma_{if} V}{v_1 + v_2}}$ Ⓜ

∴
- F.G.R. can be used to calculate "σ", the effective area presented by scaterers to each other
- This cross section "σ" is really a property of the pair of species meeting each other - rather than of either species separately.

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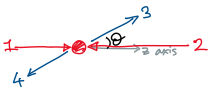
34 Let's now compute σ for the process $1+2 \to 3+4$:



‖ Aside: this is closely related to what we did on slide 28 ‖

We start by adopting Ⓔ with the replacements:

$E_1 \to E_3$
$E_2 \to E_4$
$E_i \to E_1 + E_2$
$\mathbf{p}_1 \to \mathbf{p}_3$

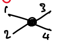
$p_2 \to p_4$

which results in:

1+2→3+4

$$\boxed{\Gamma_{if} = \underbrace{(2\pi)^4 \cdot V}_{\text{MATHS}} \cdot \int \underbrace{|T_{fi}|^2}_{\text{PHYSICS}} \underbrace{\delta(E_3+E_4-E_1-E_2)}_{\text{ENERGY CONS}} \underbrace{\delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)}_{\text{MOM CONS}} \underbrace{\frac{d^3p_3}{(2\pi)^3}\,\frac{d^3p_4}{(2\pi)^3}}_{\text{DENSITY OF STATES}}}$$ Ⓝ

Does this become nicer with the newly normalised states?

For this process $N=4$ (i.e. ~~three~~ external legs 

) and so

Ⓘ ⇒ $$M_{fi} = V^{\frac{4}{2}-1}\left(\sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_3}\sqrt{2E_4}\right)T_{fi}$$

and thus Ⓝ ⇒

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |M_{fi}|^2 \frac{V^{2(1-\frac{4}{2})}}{(2E_1)(2E_2)(2E_3)(2E_4)} \delta(E_3+E_4-E_1-E_2)\delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)\frac{d^3p_3}{(2\pi)^3}\,\frac{d^3p_4}{(2\pi)^3}$$

1+2→3+4

$$\Rightarrow \boxed{\Gamma_{if} = \frac{(2\pi)^4}{V(2E_1)(2E_2)} \int |M_{fi}|^2 \delta(E_3+E_4-E_1-E_2)\delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)\frac{d^3p_3}{(2\pi)^3\,2E_3}\,\frac{d^3p_4}{(2\pi)^3\,2E_4}}$$ Ⓞ

---

35 ∴ Ⓜ & Ⓞ ⇒

1+2→3+4

$$\boxed{\sigma = \underbrace{\frac{(2\pi)^{-2}}{(2E_1)(2E_2)(v_1+v_2)}}_{????} \int \underbrace{|M_{fi}|^2}_{\text{PHYSICS}} \underbrace{\delta(E_3+E_4-E_1-E_2)}_{\text{ENERGY CONS}} \underbrace{\delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)}_{\text{MOM CONS}} \underbrace{\frac{d^3p_3}{2E_3}\,\frac{d^3p_4}{2E_4}}_{\text{DENSITY OF STATES}}}$$ Ⓟ

↑ INVESTIGATE!

σ should* be **INVARIANT** w.r.t. Longitudinal Lorentz Boosts.

*(No Lorentz contraction ⊥ to Lorentz boost!)

(ENERGY CONS), (MOM CONS) and (PHASE-SPACE) terms are certainly Lorentz **INVARIANT.**

APPENDIX 1 (slide 45) shows that (provided $\vec{p}_1$ & $\vec{p}_2$ are collinear, which is a property preserved by Longitudinal boosts) it is always the case that

$$(2E_1)(2E_2)(v_1+v_2) = F \underset{def}{\equiv} 4\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2}$$

Hence the "????" term is Longitudinally Lorentz **INVARIANT** too!

(And as a corollary we see that consistency will thus force $|M_{fi}|^2$ to be Longitudinally Lorentz **INVARIANT** too.)*

* $|M_{fi}|^2$ is actually Lorentz INVARIANT, but the corollary on the left only guarantees this Longitudinally.

SUPER NICE

(Longitudinally) Lorentz Invariant Flux

$$F \underset{def}{\equiv} 4\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2} = (2E_1)(2E_2)(v_1+v_2)$$

is called the Lorentz Invariant Flux.

As it is Lorentz Invariant, it can be evaluated easily in different frames to get different looking (but ultimately identically valued) expressions for it. E.g:

Centre of Mass Frame	Fixed Target Frame
$F = 4E_1^* E_2^*(v_1^* + v_2^*)$	$F = 4E_1 E_2(v_1 + v_2)$
$= 4E_1^* E_2^*\left(\frac{\vert\vec{p}_1^*\vert}{E_1^*} + \frac{\vert\vec{p}_2^*\vert}{E_2^*}\right)$	$= 4E_1 m_2(v_1 + 0)$
$= 4(E_2^* + E_1^*)\vert\vec{p}^*\vert$	$= 4E_1 m_2 \frac{\vert\vec{p}_1\vert}{E_1}$
$= 4\vert\vec{p}^*\vert\sqrt{s}$ Ⓠ ↑ (Mandlestam s)	$= 4m_2\vert\vec{p}_1\vert$ Ⓡ

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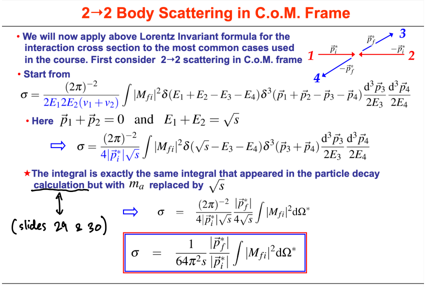

2→2 Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame

1
2
3
4
$\vec{p}_i^*$
$-\vec{p}_i^*$
$\vec{p}_f^*$
$-\vec{p}_f^*$

- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- Here $\vec{p}_1+\vec{p}_2=0$ and $E_1+E_2=\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

★The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

↕ (slides 29 & 30)

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 \mathrm{d}\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 \mathrm{d}\Omega^*}$$